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On DYNA3D's Shell Element Hourglass Constitutive Models

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1 INTRODUCTION

The explicit finite element code DYNA3D (Zywicz and Lin, 2013) contains three, 4-node shell C^0 element formulations that utilize single point in-the-plane integration with stabilization. They are the Hughes-Liu (HL) shell (Hughes and Liu, 1981; Hallquist, Benson, and Goudreau, 1985), the Belytschko, Lin, and Tsay (BLT) shell (Belytschko, Lin, and Tsay, 1984), and the membrane (MEM) element. The MEM element is a degenerate version of the BLT shell with one through-the-thickness integration point located on the mid-surface. DYNA3D contains two basic stabilization formulations for these shells – a stiffness form and a viscous form. The stabilization forms may be used separately or in combination. The stiffness formulation is the one presented by Belytschko *et al.* (1984) and is implemented in DYNA3D nearly verbatim. The viscous form, whose presence in DYNA3D predates the stiffness form, is due to Hallquist (1987). The derivation of the viscous coefficients is assumed to follow the same approach used for solid elements (Hallquist, 1987), although there is no documentation to support this. Both formulations utilize the same mechanics to generate the generalized hourglass strain rates and impose the resulting hourglass forces and moments; the formulations only differ in their hourglass constitutive idealizations.

This report serves several purposes. It develops and documents two hourglass constitutive formulations – 1) a combined linear viscous and stiffness model and 2) a linear viscous model. It calculates the frequency of each hourglass mode, assuming a rectangular geometry, for both constitutive idealizations and compares them to the nominal critical frequency of the element. This provides guidance on how to select hourglass parameters so the hourglass stabilization does not impact the time-step size and when time-step size adjustments are required due to hourglass control.

The report proceeds as follow. In Section 2, a spectral representation of a single, rectangular, linear-elastic element is developed with attention focused on the stabilization modes. A short discussion of how DYNA3D determines the critical time-step size for shell elements follows. In Section 3 the stiffness hourglass formulation presented by Belytschko et al. (1984) is summarized and its impact on the critical time-step size is explored. The theoretical framework presented here is then used to develop and analysis a combined linear viscous and stiffness hourglass constitutive model in Section 4 and a linear viscous hourglass constitutive model in Section 5. The findings are summarized and discussed in Section 6.

2 SPECTRAL REPRESENTATION

A spectral representation is used to diagonalize the equations of motion associated with a single BLT shell element. The decoupled equations associated with stabilization are analyzed to determine if and how they impact the critical time-step size of the single element. Let $\mathbf{M}^e$, $\mathbf{C}^e$, and $\mathbf{K}^e$ denote the element mass, damping, and linear stiffness matrices, respectively. $\mathbf{M}^e$ is diagonal and entries associated with translational degrees of freedom are given by

$$m_{ii}^e = \frac{\rho A t}{4} \quad \text{(no sum)} \tag{1}$$

and those associated with rotational degrees of freedom are given by

$$m_{ii}^e = \frac{\rho A t}{4}\alpha \quad \text{(no sum).} \tag{2}$$

Here t and A are the shell thickness and area, respectively, and, the material density is ρ. For rotational degrees of freedom

$$\alpha = \max\left(\frac{t^2}{12}, \frac{A}{8}\right) \tag{3}$$

when the midsurface is used as the reference surface (Hughes, 1987), and, for convenience in this report, α equals 1.0 for translational degrees of freedom. The element matrices $\mathbf{C}^e$ and $\mathbf{K}^e$ include the hourglass contributions $\mathbf{C}^{HQ}$ and $\mathbf{K}^{HQ}$, respectively. The stiffness matrix $\mathbf{K}^e$ is positive semi-definite. The damping matrix $\mathbf{C}^e$ is positive semi-definite and is constructed such that the hourglass modes are orthogonal[1] to each other, the deformation modes, and the rigid-body modes. In terms of the global position vector $\boldsymbol{x}^g$, the global equations of motion for the element are

$$\mathrm{M}_{\mathrm{ij}}^e \ddot{x}_j^g + \mathrm{C}_{\mathrm{ij}}^e \dot{x}_j^g + \mathrm{K}_{\mathrm{ij}}^e x_j^g = 0\,. \tag{4}$$

(Unless otherwise stated, repeated indices are summed.) The global equations of motion are decoupled via a spectral method that involves solving a quadratic eigenvalue problem (e.g., Tisseur and Meerbergen, 2001). For non-rigid body modes, the decoupled equation of motion for each mode is given by

$$m^m \ddot{a} + c^m \dot{a} + k^m a = 0\,. \tag{5}$$

The generalized modal displacement variable is a. The scalar quantities m^m, c^m, and k^m represent the modal mass, damping, and stiffness and equal □$^{\mathrm{T}}\mathbf{M}^{\mathrm{e}}$□, □$^{\mathrm{T}}\mathbf{C}^{\mathrm{e}}$□, and □$^{\mathrm{T}}\mathbf{K}^{\mathrm{e}}$□, respectively. Here □ denotes the eigenvector associated with the mode. When $k^m = 0$, Eq. (5) is first order in time and its time constant τ^m is given by

$$\tau^m = \frac{m^m}{c^m}\,. \tag{6}$$

When $k^m > 0$, the generalized modal equation of motion is second order in time. Its undamped natural frequency is given by

[1] Orthogonal in the quadratic eigenvalue problem sense

$$\omega^{m\,2} = \frac{k^m}{m^m}, \tag{7}$$

and the fraction of critical damping for the mode is

$$\xi^m = \frac{c^m}{2\sqrt{m^m k^m}}. \tag{8}$$

When the staggered form of explicit central difference method is used to integrate the equation of motion and the viscous force is determined using the trailing "*n-1/2*" velocity, stability is preserved when the time-step size Δt satisfies, for first order systems,

$$\Delta t^m \le 2\tau^m \tag{9}$$

(e.g., Puso, 2014), and, for the second order systems,

$$\Delta t^m \le \frac{2}{\omega^m}\left(\sqrt{1+\xi^{m\,2}} - \xi^m\right) \tag{10}$$

(Belytschko, *et al,* 2000). In the limit, as $k^m \to 0$, Eq. (10) becomes

$$\Delta t^m \le 2\frac{m^m}{c^m}, \tag{11}$$

which is the same as for the first order system.[2]

In DYNA3D the critical time-step size is based upon the deformation modes only. For an undamped shell element the critical time-step size is calculated as

$$\Delta t^e_{cr} = \frac{l}{c}. \tag{12}$$

Here l is a characteristic element length normally taken as its area divided by its longest side, and c is the sound of speed of a dilatational wave given by

$$c = \sqrt{\frac{E'}{\rho}}, \tag{13}$$

where E' is the sound-speed modulus. For an isotropic elastic material $E' = 2G + \lambda$, where λ is the Lame constant, and for materials with a non-negative Poisson's ratio (ν) $E' \ge E$. The corresponding frequency to Δt^e_{cr} is ω^e_{cr} which equals $2c/l$. For a rectangular geometry, the element length l can be bounded from above in terms of the gradient operator matrix **B**, which is defined in Section 3, as

$$l \le \frac{\sqrt{2}}{\|B\|}. \tag{14}$$

Thus,

$$\Delta t^e_{cr} \le \frac{\sqrt{2}}{c\|B\|}. \tag{15}$$

Let R define the ratio of Δt^e_{cr} to the time-step size of the mode of interest, Δt^i, as

[2] When $k^m = 0$, the integration algorithm becomes an explicit forward Euler method with respect to the mid-step velocity.

$$R \equiv \frac{\Delta t_{cr}^{e}}{\Delta t^{i}}. \tag{16}$$

When $R \le 1$ the mode does not influence the element time-step size. Let $\bar{R}$ be an upper bound estimate of R formed using the upper bound estimate for l. For a first order system define $\bar{R}$ as

$$\bar{R} \equiv \frac{\sqrt{2}}{2c\|B\|\tau^{i}} \tag{17}$$

and for a second order system as

$$\bar{R} \equiv \frac{\sqrt{2}\,\omega^{i}}{2c\|B\|\left(\sqrt{1+\xi^{i2}}-\xi^{i}\right)}. \tag{18}$$

When the condition $\bar{R} \le 1$ is satisfied so is the condition that $R \le 1$.

3 STIFFNESS HOURGLASS CONTROL

The stiffness-based stabilization developed and presented by Belytschko *et al.* (1984) is summarized as follows. A *x-y-z* corotational coordinate system is embedded in the shell. The midsurface of the element, which is assumed flat, defines the *x-y* plane, and, without loss of generality, the coordinate origin coincides with node 1. The coordinates, translational velocities, and rotational velocities of node I are given by (x_I,y_I,z_I), (v_{xI},v_{yI},v_{zI}), and ($\dot{\theta}_{xI}$,$\dot{\theta}_{yI}$,$\dot{\theta}_{zI}$), respectively. (Unless otherwise stated, I spans from 1 to 4.) The components of the gradient operator **B** matrix are given by

$$B_{1I} = \frac{1}{2A}\left[y_2 - y_4, y_3, y_4 - y_2, -y_3\right] \tag{19}$$

$$B_{2I} = \frac{1}{2A}\left[y_2 - y_4, y_3, y_4 - y_2, -y_3\right], \tag{20}$$

where A is the area of the element. To calculate the generalized hourglass strain rate, the matrix $\boldsymbol{\gamma}$ is defined as

$$\gamma_I = h_I - \left[\left(h_J x_J\right)B_{1I} + \left(h_K y_K\right)B_{2I}\right] \tag{21}$$

Here J and K span from 1 to 4. The vector **h** is defined as

$$\mathbf{h} = [+1, -1, +1, -1], \tag{22}$$

and, independent of the element geometry, $\gamma_I h_I = 4$. The generalized hourglass strain rates are expressed as

$$\dot{q}_1^{b} = \gamma_I \dot{\theta}_{xI} \text{ and } \dot{q}_2^{b} = \gamma_I \dot{\theta}_{yI}, \tag{23}$$

$$\dot{q}_3^{w} = \gamma_I \dot{v}_{zI}, \tag{24}$$

$$\dot{q}_4^{m} = \gamma_I \dot{v}_{xI} \text{ and } \dot{q}_5^{m} = \gamma_I \dot{v}_{yI}, \tag{25}$$

where the superscripts *b, w,* and *m* designate the hourglass modes associated with bending, W-mode (out-of-plane), and membrane (in-plane), respectively. For stiffness hourglass control, the generalized hourglass strain rates are related to the generalized hourglass stress rates $\dot{Q}_i$, *i.e.*, the hourglass constitutive relationship, via

$$\dot{Q}_1^{b} = C_1 \dot{q}_1^{b} \text{ and } \dot{Q}_2^{b} = C_1 \dot{q}_2^{b}, \tag{26}$$

$$\dot{Q}_3^w = C_2 \dot{q}_3^w, \tag{27}$$

$$\dot{Q}_4^m = C_3 \dot{q}_4^m \text{ and } \dot{Q}_5^m = C_3 \dot{q}_5^m, \tag{28}$$

where $C_1 = \frac{r_b}{192} E t^3 A \|B\|^2$, $C_2 = \frac{r_w}{12} \kappa G t^3 \|B\|^2$, and $C_3 = \frac{r_m}{8} E t A \|B\|^2$. The norm of the **B** matrix is defined as $\|B\| = \sqrt{B_{\beta I} B_{\beta I}}$, where β spans 1 to 2. The quantities E, G, κ, and t are the Young's modulus, shear modulus, shear correction factor, and thickness of the shell element, respectively. The parameters r_b, r_w, and r_m scale the magnitude of the hourglass stresses and typically have values between 0.01 and 0.05. The hourglass stresses are found by directly integrating the stress rate in time as

$$Q_i = \int \dot{Q}_i dt. \tag{29}$$

Finally, the generalized nodal forces and moments due to the hourglass stresses are

$$m_{xI}^b = \gamma_I Q_1^b \text{ and } m_{yI}^b = \gamma_I Q_2^b, \tag{30}$$

$$f_{zI}^w = \gamma_I Q_3^w, \tag{31}$$

$$f_{xI}^m = \gamma_I Q_4^m \text{ and } f_{yI}^m = \gamma_I Q_5^m. \tag{32}$$

The spectral representation is now applied to a single, rectangular, BLT shell. The eigenvector □ for each hourglass mode is defined such that the entries associated with the participating degrees of freedom mirror **h** and all other entries are zero. For the mass, one finds m^m equals $\rho A t \alpha$. When the non-participating degrees of freedom are excluded, the stiffness sub-matrix for each stabilization mode has the form

$$K_{jk}^{HQ} = C \gamma_j \gamma_k. \tag{33}$$

Consequently, c^m and k^m equal 0 and 16C, respectively.

The frequency for each stiffness hourglass mode is now determined along with its time-step ratio $\bar{R}$. The frequencies for the bending modes are

$$\omega^b = \|B\| \sqrt{\frac{r_b E}{\rho}}, \text{ when } \frac{t^2}{12} \geq \frac{A}{8} \tag{34}$$

and

$$\omega^b = \frac{t}{\sqrt{A}} \|B\| \sqrt{\frac{2}{3} \frac{r_b E}{\rho}}, \text{ otherwise.} \tag{35}$$

For the W-mode the frequency is

$$\omega^w = \frac{t}{\sqrt{A}} 2 \|B\| \sqrt{\frac{r_w \kappa G}{3\rho}}, \tag{36}$$

and the frequency for the membrane modes is

$$\omega^m = \|B\| \sqrt{\frac{2 r_m E}{\rho}}. \tag{37}$$

Based on Eq. (18), the requirements for these modes not to influence the critical time-step size of the element are

$$\text{Bending Modes: } \bar{R}^b = \frac{\sqrt{2}}{2}\sqrt{r_b \frac{E}{E'}} \le 1, \text{ when } \frac{t^2}{12} \ge \frac{A}{8} \tag{38}$$

$$\bar{R}^b = \frac{t}{\sqrt{A}}\sqrt{\frac{r_b}{3}\frac{E}{E'}} \le 1, \quad \text{otherwise} \tag{39}$$

:

$$\text{W-mode: } \bar{R}^w = \frac{t}{\sqrt{A}}\sqrt{\frac{r_m \kappa}{3(1+\nu)}\frac{E}{E'}} \le 1 \tag{40}$$

$$\text{Membrane Modes: } \bar{R}^m = \sqrt{r^m \frac{E}{E'}} \le 1. \tag{41}$$

For the bending mode, Eq. (38) provides an upper bound to Eq. (39). Hence, satisfying Eq. (38) for all thickness-to-area ratios ensures that Eq. (39) is also satisfied when it is applicable.

Based upon the time-step ratios, neither the bending modes nor the membrane mode will control the time-step size for a rectangular shell element of any aspect ratio provided that r_b and r_m are selected such that Eqs. (38) and (40) are satisfied. This is not the case for the W-mode. For any value of r_w, there is an associated critical value of $t/\sqrt{A}$ that, if exceeded, will cause $R^w > 1$. Luckily, for typical engineering problems and common values of r_w, this only occurs when $t > \sqrt{A}$, i.e., for "stubby" elements. For example, when $r_w = 0.05$, $\kappa = 5/6$, and $\nu = 0.3$ for an isotropic elastic material, the requirement in Eq. (40) becomes

$$t \le 5.683\sqrt{A}. \tag{42}$$

Clearly, attention needs to be paid to the value of r_w and/or the element critical time-step size when $t > \sqrt{A}$.

4 COMBINED STIFFNESS AND VISCOUS HOURGLASS CONTROL

A combined stiffness and viscous hourglass constitutive relationship is developed, and its time-step size ramifications are examined. Motivated by Rayleigh stiffness proportional damping, define the hourglass constitutive relationship as

$$Q = \eta \dot{q} + \int C \dot{q}\, dt \tag{43}$$

and, when non-participating degrees of freedom are excluded, the damping sub-matrix for each stabilization mode as

$$C_{jk}^{HQ} = \eta \gamma_j \gamma_k. \tag{44}$$

Here η is the viscosity. To facilitate the linear stability analysis, idealize the constitutive relationship as

$$Q = \eta \dot{q} + Cq, \tag{45}$$

where

$$q = \int \dot{q}\, dt. \tag{46}$$

Application of the spectral decomposition yields the hourglass modal coefficients m^m, c^m, and k^m whose values are $\rho A t \alpha$, 16η, and $16C$, respectively. From Eq. (8), the fraction of critical damping is given by

$$\xi = \frac{2\eta}{\sqrt{\rho A t \alpha\, C}}. \tag{47}$$

Using this, the viscosity may now be expressed as

$$\eta = \frac{\xi}{2}\sqrt{\rho A t \alpha\, C} \tag{48}$$

so that Eqs. (43) and (45) become

$$Q = \frac{\xi}{2}\sqrt{\rho A t \alpha\, C}\,\dot{q} + \int C\dot{q}\, dt \tag{49}$$

and

$$Q = \frac{\xi}{2}\sqrt{\rho A t \alpha\, C}\,\dot{q} + C q, \tag{50}$$

respectively.

When this hourglass constitutive idealization is used, the equations of motion for each hourglass mode are second order. From Eq. (7) the modal frequency is

$$\omega = \sqrt{\frac{16C}{\rho A t\,\alpha}}, \tag{51}$$

and from Eq. (17)

$$\bar{R} = \frac{2\sqrt{2}}{\sqrt{1+\xi^2} - \xi}\sqrt{\frac{gr}{\alpha}\frac{E}{E'}}, \tag{52}$$

where

$$g = \frac{C}{rtA\,E\|B\|^2}. \tag{53}$$

Let ξ_{cr} denote the critical value of ξ at which $\bar{R} = 1$. From Eq. (52) ξ_{cr} is found to be

$$\xi_{cr} = \sqrt{\frac{\alpha}{32rg}\frac{E'}{E}} - \sqrt{\frac{2rg}{\alpha}\frac{E'}{E}}. \tag{54}$$

The critical viscosity η_{cr} is found by inserting ξ_{cr} into Eq. (47), which after some manipulation, yields

$$\eta_{cr} = \frac{At\|B\|}{\sqrt{2}}\left(\frac{\rho c\alpha}{8} - \frac{rgE}{c}\right) \tag{55}$$

or, since l is represented by $\sqrt{2}/\|B\|$,

$$\eta_{cr} = \frac{At}{l}\left(\frac{\rho c\alpha}{8} - \frac{rgE}{c}\right). \tag{56}$$

When $\xi \le \xi_{cr}$ or, equivalently, $\eta \le \eta_{cr}$, the hourglass mode will not alter the critical time-step size since $\bar{R} \le 1$. For the three different types of hourglass modes, the maximum values of ξ and η are:

$$\text{Bending Mode: } \xi^b \le \sqrt{\frac{6}{r_b}\frac{\alpha}{t^2}\frac{E'}{E}} - \sqrt{\frac{r_b}{96}\frac{t^2}{\alpha}\frac{E}{E'}} \tag{57}$$

$$\text{W-Mode: } \xi^w \le \sqrt{\frac{3(1+\nu)}{4r_w\kappa}\frac{A}{t^2}\frac{E'}{E}} - \sqrt{\frac{r_w\kappa}{12(1+\nu)}\frac{t^2}{A}\frac{E}{E'}} \tag{58}$$

$$\text{Membrane Mode: } \xi^m \le \frac{1}{2}\left(\sqrt{\frac{1}{r_m}\frac{E'}{E}} - \sqrt{r_m\frac{E}{E'}}\right) \tag{59}$$

and

$$\text{Bending mode: } \eta^b_{cr} = \frac{At}{8l}\rho c\alpha\,\zeta^b \text{ and } \zeta^b = 1 - r_b\frac{t^2}{24\alpha}\frac{E}{E'} \tag{60}$$

$$\text{W-mode: } \eta^w_{cr} = \frac{At}{8l}\rho c\,\zeta^w \text{ and } \zeta^w = 1 - r_w\frac{r_w\kappa}{3(1+\nu)}\frac{E}{E'}\frac{t^2}{A} \tag{61}$$

$$\text{Membrane mode: } \eta^m_{cr} = \frac{At}{8l}\rho c\,\zeta^m \text{ and } \zeta^m = 1 - r_m\frac{E}{E'}. \tag{62}$$

The maximum modal viscosities are very similar. For typical engineering materials and scaling parameters ($r \approx 0.10$), the value of ζ is about 0.9 (for non-stubby elements). Let $\tilde{\eta}_{cr}$ denote the value of η_{cr} when $\zeta = 1$ as:

$$\tilde{\eta}^b_{cr} = \frac{At}{8l}\rho c\alpha, \quad \tilde{\eta}^w_{cr} = \frac{At}{8l}\rho c, \quad \text{and} \quad \tilde{\eta}^m_{cr} = \frac{At}{8l}\rho c. \tag{63}$$

In the absence of any stiffness stabilization, $\tilde{\eta}_{cr}$ represents the maximum viscosity that can be used with no time-step size ramifications and allows the viscosity in Eqs. (60-62) to be written as

$$\eta = \tilde{r}\eta_{cr}, \tag{64}$$

where $\tilde{r}$ is the viscous hourglass scaling parameter.

While general proportional damping has motivated this derivation, it is not convenient to define the constitutive parameters in this manner since ξ varies with *r*. A more practical way to express the constitutive model is as

$$Q = \tilde{r}\tilde{\eta}_{cr}\,\dot{q} + \int C\dot{q}\,dt. \tag{65}$$

In this form, once can scale the viscous and stiffness contributions independently. Provided that $0 \le r \le 0.10$ and $0 \le \tilde{r} \le 1$ (or, formally, $0 \le \tilde{r} \le 1 - rE/E'$), the hourglass stabilization will not affect the time-step size for non-stubby elements. Observe, when *r* equals zero a pure viscous hourglass constitutive relationship emerges.

5 VISCOUS HOURGLASS CONTROL

Consider a viscous hourglass constitutive relationship given by

$$Q = \eta\dot{q}, \tag{66}$$

and whose damping sub-matrix is defined by Eq. (44). Application of the spectral decomposition to a single BLT shell employing this hourglass constitutive model yields the hourglass modal coefficients m^m, c^m, and k^m values as $\rho At\alpha$, 16η, and 0, respectively. The resulting equations of motion for each hourglass mode are first order in time. The time coefficient is given by

$$\tau = \frac{\rho A t \alpha}{16\eta}, \tag{67}$$

and, the time-step ratio $\bar{R}$ is expressed as

$$\bar{R} = \frac{l}{c}\frac{8\eta}{\rho A t \alpha}. \tag{68}$$

For the hourglass stabilization not to alter the element time-step size $\bar{R} \leq 1$ or, equivalently,

$$\eta \leq \frac{\rho c}{8l} A t \alpha. \tag{69}$$

Therefore for the three mode types, the maximum permitted viscosities are:

$$\text{Bending mode: } \eta_{cr}^{b} = \frac{\rho c}{8l} A t \alpha \tag{70}$$

$$\text{W-mode: } \eta_{cr}^{w} = \frac{\rho c}{8l} A t \tag{71}$$

$$\text{Membrane mode: } \eta_{cr}^{m} = \frac{\rho c}{8l} A t \tag{72}$$

A more convenient way to define the constitutive relation, i.e., Eq. (66), is by

$$Q = \tilde{r}\eta_{cr}\dot{q}, \tag{73}$$

where $\tilde{r}$ is the viscous scaling parameter. Provided that $0 \leq \tilde{r} \leq 1$, the hourglass modes will not affect the element time-step size.

The constitutive relationship defined by Eq. (73) and the viscosities given by Eqs. (70-72) are the same as those derived in section 4, Eq. (65) and Eqs. (60-62), when the stiffness scaling parameter *r* was set equal to zero.

6 SUMMARY AND DISCUSSION

Two hourglass constitutive models were formulated for use within the stabilization framework developed by Belytschko, *et al.* (1984). A viscous model and a combined viscous and stiffness hourglass model were formulated. In the limit, when the stiffness contribution in the combined model went to zero, the viscous model was recovered.

The original stiffness constitutive model and the two new constitutive models were investigated to determine if and how they affect the critical time-step size of a single, rectangular, BLT shell. When reasonable values for the stiffness scaling parameter *r* are used, $r \leq 0.10$, stiffness-based stabilization does not, in general, negatively impact the critical time-step size for elements whose thickness is less than or equal to its in-plane dimension. For stubby elements, those whose thickness is greater than approximately five times their characteristic in-plane dimension, the frequency of the W-mode can exceed the estimated highest frequency of the element. Consequently, either a smaller scaling parameter or integration time-step size must be used to maintain time-integration stability. These results are also true for combined viscous and stiffness stabilization when the viscosities do not exceed their critical values. For the viscous only model, the time-step size is unaffected, even for stubby elements, provided that the viscous scaling parameter $\tilde{r}$ does not exceed one.

In DYNA3D, the input stiffness and viscous hourglass scaling parameters are *hgq_stiff* and *hgq_vis*, respectively. If neither of these quantities is explicitly defined, then they are set based upon the value of *HQ*, whose default value is 0.10, and the hourglass constitutive type as:

$$\text{Stiffness stabilization: } hgq\_stiff = HQ \;\text{ and }\; hgq\_vis = 0$$

$$\text{Viscous stabilization: } hgq\_stiff = 0 \text{ and } hgq\_vis = HQ$$

$$\text{Combined stiffness \& viscous: } hgq\_stiff = 0.95\,HQ \;\text{ and }\; hgq\_vis = 0.05\,HQ$$

The scaling parameters used in this report are related to *hgq_stiff* and *hgq_vis* by

$$r = \frac{hgq\_stiff}{40} \tag{74}$$

and

$$\tilde{r} = 2\,hgq\_vis\,. \tag{75}$$

Thus, when default values are used for pure stiffness or pure viscous stabilization r equals 0.0025 with $\tilde{r}$ equals zero, and r equals zero with $\tilde{r}$ equals 0.2, respectively. Consequently, the default values do not alter the critical time-step size of the element for non-stubby elements. When the combined stabilization from is used, both *hgq_stiff* and *hgq_*vis can be set to 0.1 without causing any time-step size issues.

In most engineering problems, the hourglass modes are not perfectly orthogonal to the deformation modes, and the critical time-step for the analysis is actually greater than the individual element estimates. Consequently, one may chose to approximate certain quantities for computational convenience without adverse consequences especially when sub-critical hourglass parameters are used. For example, in Eqs. (60-62) and (70-72), A/l might be approximated by $\sqrt{A}$.

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